Damped pendulum with external forcing function

1 Introduction

A damped pendulum subjected to an external (harmonic) excitation function $e(t)$ exhibits chaotic behavior and can be easily simulated on an analog computer as the following application note shows.[1]

We assume a punctiform mass m on a massless pendulum rod of length r. The angle between this rod and a vertical line crossing its hinge is denoted by φ. g is the acceleration of gravity, β represents the damping factor, while A is the amplitude of the excitation function.

With I denoting the rotational inertia, we have

$$I = mr^2.$$

Since the sum of all forces in a closed system must be zero, it follows that

$$I\ddot{\varphi} = \sum_i \tau_i, \tag{1}$$

where the τ_i denote the various torques acting in the system. These are

$\tau_g = -rmg\sin(\varphi)$	torque due to g
$\tau_\beta = -\beta\dot{\varphi}$	torque due to β
$\tau_e = A\cos(\omega t)$	torque due to excitation

[1]This note was motivated by https://www.myphysicslab.com/pendulum/pendulum/chaotic-pendulum-en.html.

Applying these to (1) results in

$$mr\ddot{\varphi} = -rmg\sin(\varphi) - \beta\dot{\varphi} + A\cos(\omega t)$$

which can be rearranged to

$$\ddot{\varphi} = -\frac{g}{r}\sin(\varphi) + \frac{A\cos(\omega t) - \beta\dot{\varphi}}{mr^2}.$$

As the highest derivative is isolated on the left-hand side this form is ideally suited to derive an analog computer setup by KELVIN's feedback technique.

2 Excitation function

First of all we need a harmonic excitation function with a variable frequency. Basically such a problem can always be reduced to solving a second order ordinary differential equation of the form $\ddot{y} = -y$. Due to unavoidable imperfections of real integrators we will add some amplitude stabilization as shown in figure 1.

3 Sine function

If we allow large displacement angles, the sine-term in τ_g can no longer be approximated by $\sin(\varphi) \approx \varphi$ as it is normally done in school. Instead we need a real sine generator. A sine function could be implemented by a diode function generator, approximating the sine by a polygon with rather good accuracy but this approach will fail as soon as the angle leaves the interval $[-\pi : \pi]$ (or any other fixed interval on which the

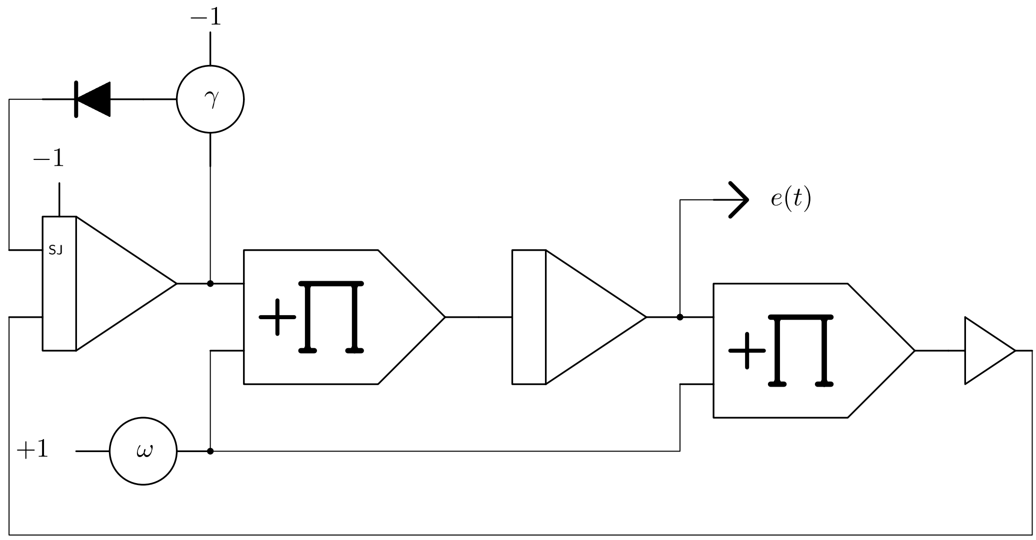

Figure 1: Forcing function $e(t)$

function generator will yield useful results) which will invariable happen when our pendulum rolls over due to the external forcing function. Thus we resort to a classic trick: Generating the sine not based on φ but rather on $\dot{\varphi}$ instead.

This makes it possible to use a second circuit of the form shown in figure 1. The two multipliers will now not be connected to a manual potentiometer but instead to the value $\dot{\varphi}$ which is readily available from the pendulum subcircuit as shown in figure 2. The output (corresponding to $e(t)$ in figure 1) then yields the value $\sin\left(\int\dot{\varphi}\,\mathrm{d}t\right)$. So all in all we need two instances of the circuit shown in figure 1.

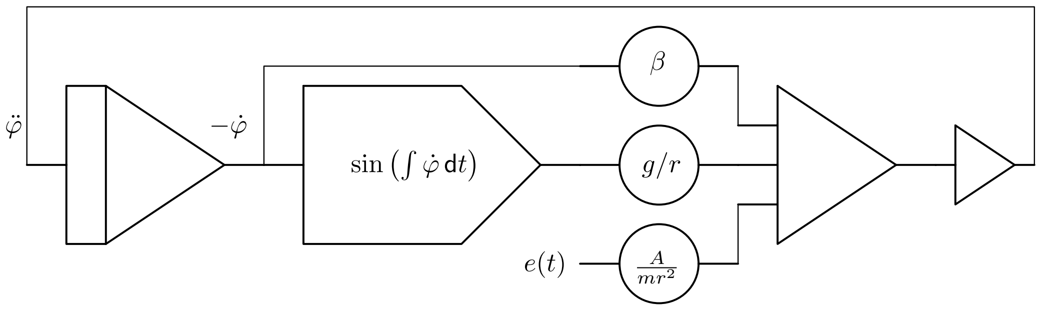

Figure 2: Computer setup for pendulum simulation

4 Pendulum simulation

Using these two subcircuits we can now setup the pendulum simulation itself as shown in figure 2.

5 Results

This setup invites to play with the parameters, β, A/mr^2, g/r, and ω. The oscilloscope screen image shown in figure 3 is a typical phase-space plot of $\ddot{\varphi}$ against $-\dot{\varphi}$.

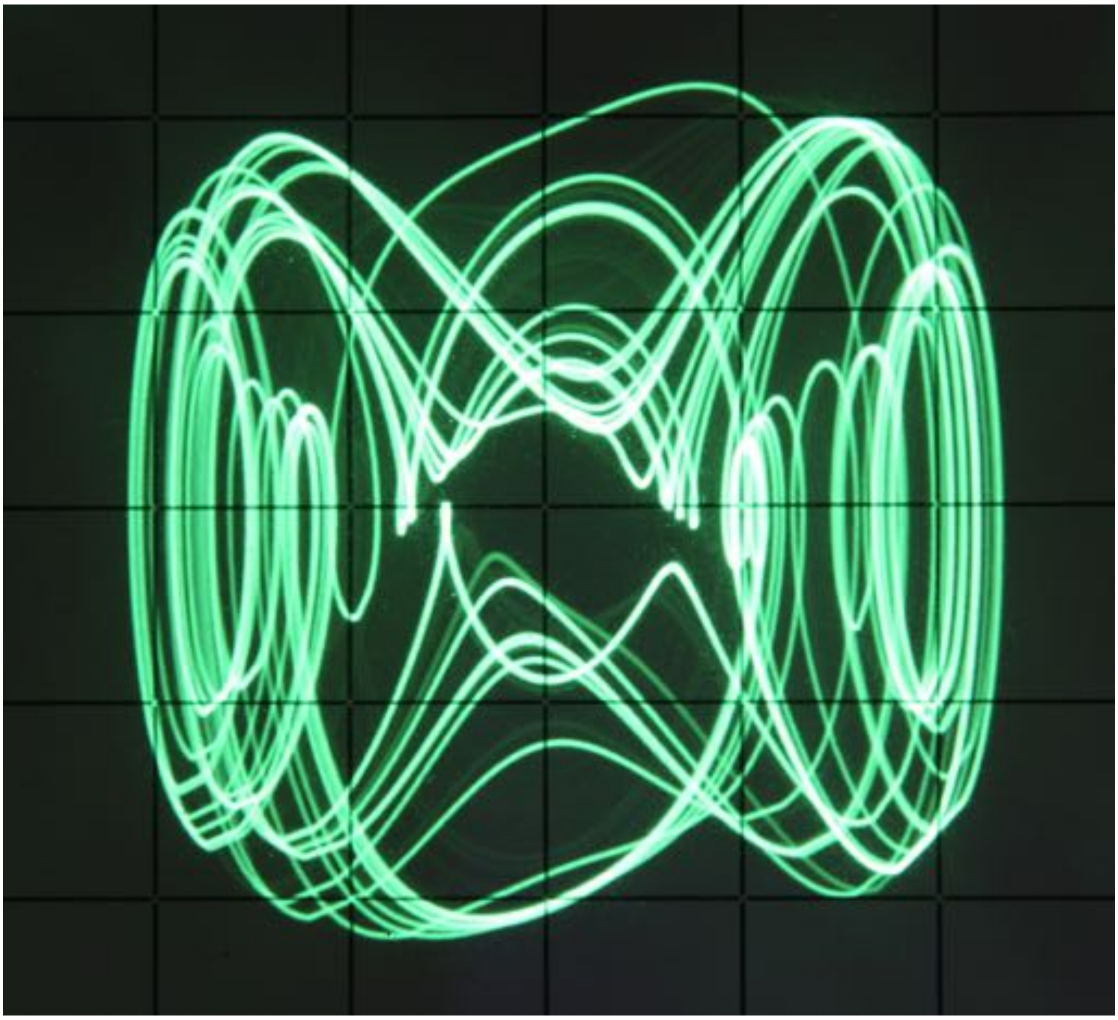

Figure 3: Typical simulation result showing chaotic behavior