A 4D hyperchaotic system

Hyperchaotic systems are dynamical systems exhibiting two positive Lyapunov exponents while classic chaotic systems only have one positive Lyapunov exponent.[1] These systems require (at least) four coupled ordinary differential equations (*ODE*s)[2] and are often hard to scale properly in order to implement them on an analog computer. The first hyperchaotic system was suggested in 1979 by Otto Rössler of Rössler attractor fame[3] and is of the form[4]

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + 0.25y + w\\ \dot{z} &= 3 + xz\\ \dot{w} &= -0.5z + 0.05w.\end{aligned}$$

In 2024 Benkouider et al.[5] suggested a novel hyperchaotic system described by

$$\begin{aligned}\dot{x} &= ax - yz + w\\ \dot{y} &= xz - by\\ \dot{z} &= xy - cz\\ \dot{w} &= -y - d\end{aligned}$$

with $a = 8$, $b = 40$, $c = 15$, and $d = 0.1$. Its Lyapunov exponents are given as $L_1 = 3.036$, $L_2 = 0.01$, $L_3 = 0$, and $L_4 = -50.032$ clearly showing the hyperchaotic characteristic of the system with $L_1 > 0$ and $L_2 > 0$.

[1] See [Jordan et al. 1999, ch. 8] for a thorough discussion of stability including Lyapunov stability, etc.
[2] In contrast to this, ime discrete systems only require three dimensions to exhibit hyperchaotic behavior.
[3] See [Rössler 1976] and the very first application note in this series, [Ulmann 2016].
[4] See [Rössler 1979].
[5] See [Benkouider et al. 2024]

A quick numeric experiment shows that suitable scaling factors are $\lambda_x = \lambda_y = \lambda_z = \frac{1}{200}$ and $\lambda_w = \frac{1}{50}$. Applying these yields $a = 8$, $b = 40$, $c = 15$, $d^* = 0.002$, and

$$\dot{x} = ax - 200yz + 0.25w \tag{1}$$
$$\dot{y} = -by + 200xz \tag{2}$$
$$\dot{z} = -cz + 200xy \tag{3}$$
$$\dot{w} = -4y - d^*. \tag{4}$$

This is not too well suited for implementation on an analog computer due to the large coefficients of 200 in equations (1), (2), and (3). This can be mitigated by scaling down all four equations by a common scaling factor of $\frac{1}{20}$ (this is effectively time scaling) yielding $a^* = 0.4$, $b^* = 2$, $c^* = 0.75$, $d^* = 0.0001$, and

$$\begin{aligned}
\dot{x} &= a^*x - 10yz + 0.0125w \\
\dot{y} &= -b^*y + 10xz \\
\dot{z} &= -c^*z + 10xy \\
\dot{w} &= -0.2y - d^*.
\end{aligned} \tag{5}$$

Unfortunately, this results in another problem: $d^* = 10^{-4}$ is not suitable for an analog computer. On a system with ± 10 V machine unit that would be just one mV and would require at least two coefficient potentiometer in series. To solve this problem, the w-integrator can be run with a much smaller time scaling factor than the other three integrators. Using $k_0 = 10^3$ for the integrators yielding $-x$, $-y$, $-z$, and $k_0 = 10$ for the $-w$-integrator changes equation (5) to

$$\dot{w} = -20y - d^\dagger \tag{6}$$

with $d^\dagger = 100d^* = 0.01$ which is well within reach of any analog computer as is the input weight of 20 in equation (6). Suitable initial conditions for this system have been determined experimentally as $x(0) = y(0) = z(0) = w(0) = -0.1$. The overall schematic is shown in figure 1.[6] The resulting phase space plots are shown in figure 2.

[6] Since three products, xy, xz, and $-yz$, are required, this program exceeds the capabilities of a single THE ANALOG THING.

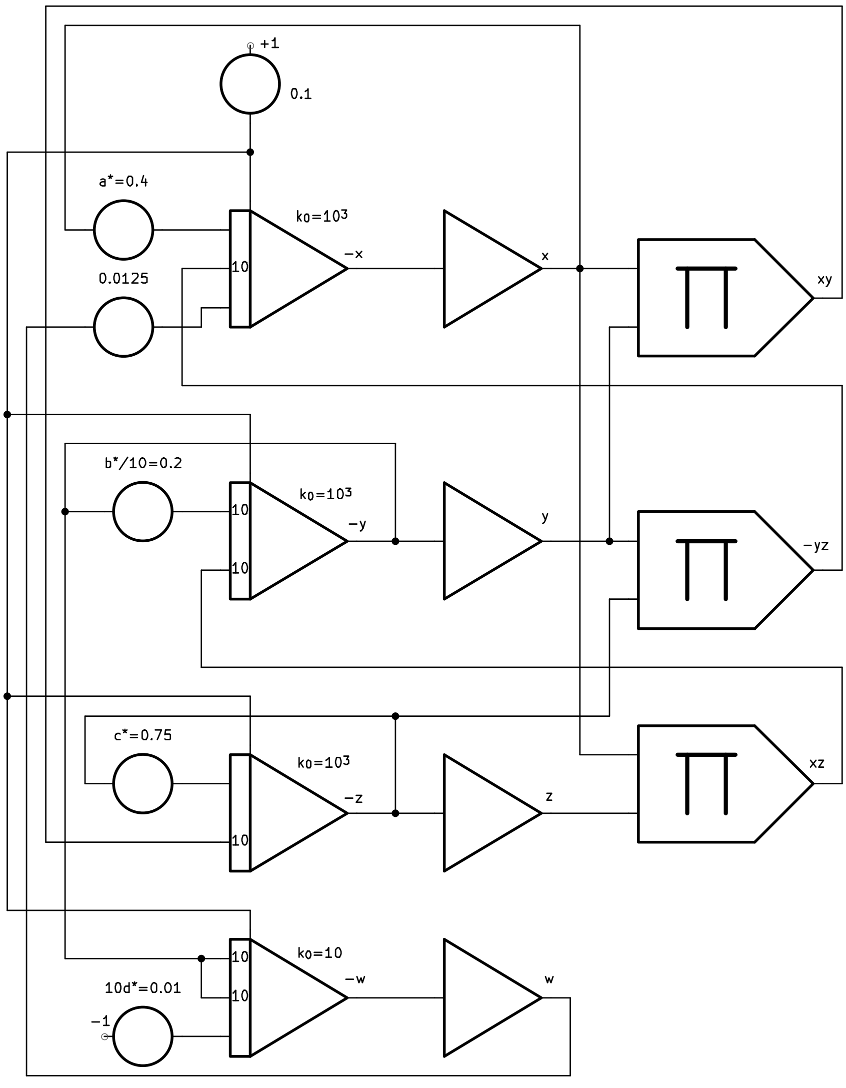

Figure 1: Implementation of the 4D hyperchaotic system

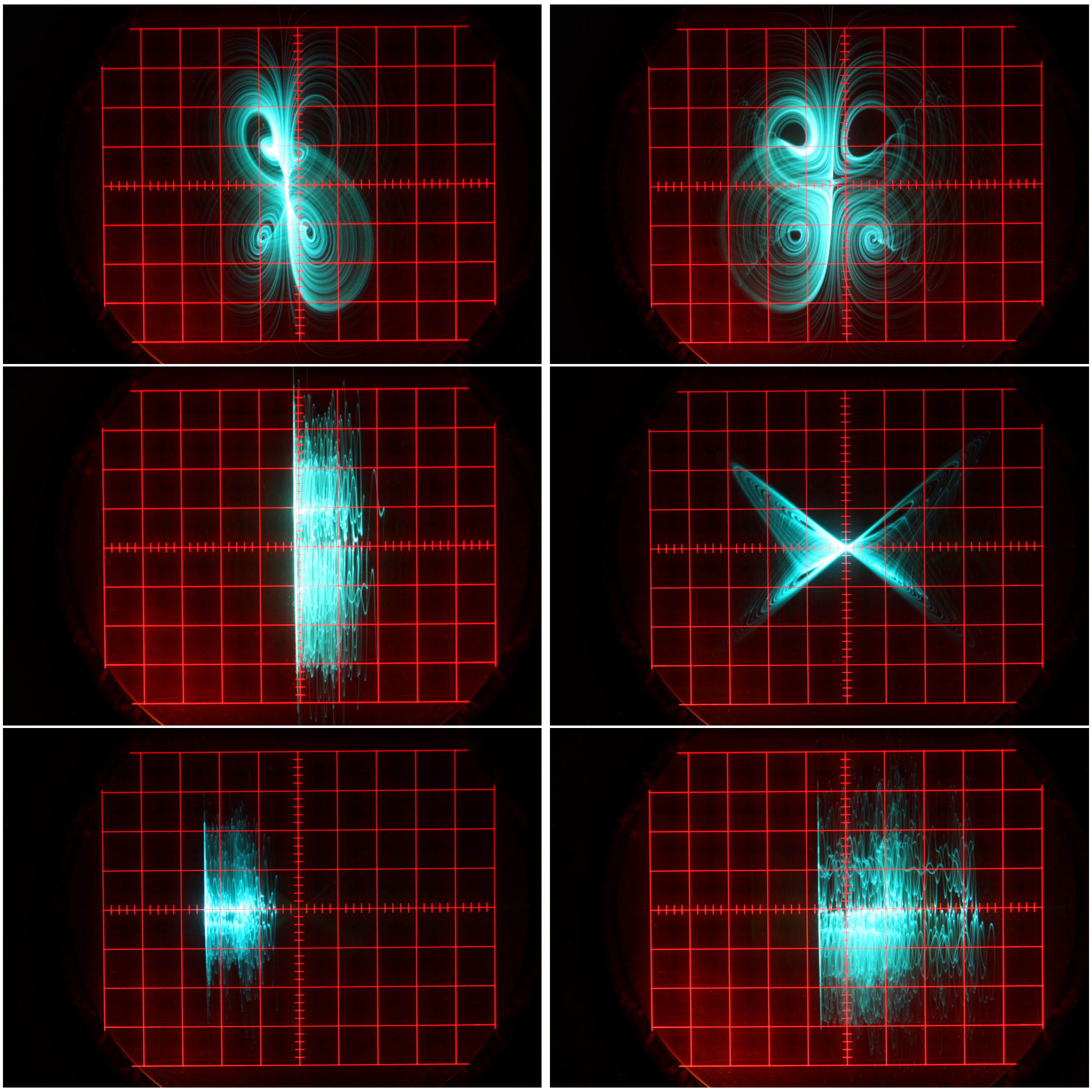

Figure 2: Phase space plots of the 4D hyperchaotic system – from left to right and top to bottom x/y, x/z, x/w, y/z, y/w, and z/w are shown

References

[Benkouider et al. 2024] Khaled Benkouider, Aceng Sambas, Talal Bonny, Wafaa Al Nassan, Issam A. R. Moghrabi, Ibrahim Mohammed Sulaiman, Basim A. Hassan, Mustafa Mamat, "A comprehensive study of the novel 4D hyperchaotic system with self-excited multistability and application in the voice encryption", in Nature, scientific reports, https://doi.org/10.1038/s41598-024-63779-1

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[Rössler 1976] Otto E. Rössler, "An Equation for Continuous Chaos", in *Physics Letters*, Volume 57A, number 5, 1976, pp. 397–398

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