Thomas' Cyclically Symmetric Attractors[1]

René Thomas and Otto E. Rössler (of "Rössler attractor" fame) once considered systems of coupled differential equations of the general form[2]

$$\dot{x_i} = -bx_i + f(x_{i+1}) \text{ mod } n \text{ with } 1 \le i \le n, n \in \mathbb{N}.$$

A particular example of this is discussed in [Thomas 1999, p. 1899] and exhibits chaotic behavior ($n = 3$) with an especially pretty phase space plot:

$$\dot{x} = -bx + \sin(y) \tag{1}$$
$$\dot{y} = -by + \sin(z) \tag{2}$$
$$\dot{z} = -bz + \sin(x) \tag{3}$$

The central parameter b controls the overall behavior of the system with $b = 0.19$ resulting in a stable periodic orbit, a chaotic attractor for $b = 0.18$, and two chaotic attractors for $b = 0.2$. Setting $b = 0$ results in an interesting chaotic system without an attractor.[3]

Scaling this system is quite easy thanks to its inherent symmetry with $\lambda = \lambda_x = \lambda_y = \lambda_z = \frac{1}{5}$.[4] This transforms equations (1), (2), and (3) into the following setup of coupled differential equations:

$$\dot{x} = \frac{-5bx + \sin(5y)}{5} = -bx + 0.2\sin(5y) \tag{4}$$
$$\dot{y} = \frac{-5by + \sin(5z)}{5} = -by + 0.2\sin(5z) \tag{5}$$
$$\dot{z} = \frac{-5bz + \sin(5x)}{5} = -bz + 0.2\sin(5x) \tag{6}$$

[1]The author would like to thank Michael Koch for pointing out several typos and other errors.
[2]See [Basios et al. 2018].
[3]Maybe this could be used as a noise source in some applications.
[4]These are scaling factors for the variables x, y, and z. They correspond to the value interval for each variable. In this example, x, y, and z are all well within $[-5, 5]$, so scaling them down by $\frac{1}{5}$ will avoid overloads in the computation. To scale a system of equations, every variable is scaled by multiplying it with its corresponding λ where it originates from while it is scaled with $\frac{1}{\lambda}$ wherever it is used. Typically most of these factors cancel out eventually, yielding a system of scaled equations suitable for implementation on an analog computer.

As simple as these equations look their actual implementation on an analog computer poses a problem due to the sine terms. First of all, the argument of these functions will be in the range of roughly ± 5 radians. Second, there are no readily available sine function generators on the market since the fascinating AD639 integrated circuit was phased out of production long ago.

Since the equations yield the first time derivatives of x, y, and z, the required sine terms can be generated solving auxiliary differential equations. That is a standard technique outlined in [GILOI et al. 1963, p. 187 f.] or [KORN et al. 1964, 8-16]. What is required are functions of the form $u = \sin(x)$, etc.[5] Differentiating this twice yields the auxiliary equation

$$\ddot{u} = -\dot{x}^2 \sin(x) = -\dot{x}^2 u,$$

which describes a harmonic oscillator with variable angular frequency controlled by $\dot{x}$. These functions

$$s_x = -\frac{1}{\lambda^2}\dot{x}s_x = -25\dot{x}s_x \tag{7}$$

$$s_y = -\frac{1}{\lambda^2}\dot{y}s_x = -25\dot{y}s_y \tag{8}$$

$$s_z = -\frac{1}{\lambda^2}\dot{z}s_x = -25\dot{z}s_z \tag{9}$$

can be easily implemented by means of two integrators, two multipliers (fed with $\dot{x}$, $\dot{y}$, and $\dot{z}$, respectively) and an inverter for each of the required three sine functions.

Since $\sin(x)$, etc., are required, the initial conditions for these equations must be chosen accordingly to satisfy $\sin(x(0))$, etc. Unfortunately, $x(0) = y(0) = z(0) = 0$ will not work since the system will stay at the origin and not evolve, but choosing a small value such as $x(0) = 0.01$ is sufficient to let the system evolve while being close enough to 0 so as to not require additional thought regarding the initial conditions for s_x, etc.

Implementing equations (4), (5), and (6) using a summer requires the signs of the terms on the right-hand side to be inverted. This is easy in the case of the bx, by, and bz terms since the integrators operating on $\dot{x}$, $\dot{y}$, and $\dot{z}$ perform an implicit sign inversion as well.

[5] As is customary, the argument t is omitted for clarity in the following, x is to be understood as $x(t)$.

However, it would require three additional inverters yielding $-s_x$, $-s_y$, and $-s_z$. These can be avoided by changing the sign of the initial condition(s) of these three terms, thus saving three inverters.

The actual implementation of the sine function generators is a bit nasty due to the unavoidable imperfections of real integrators. Without additional means for amplitude stabilization, the amplitude will typically rise well above ± 1 for high(er) frequencies while it will decrease when the derivative input is near zero. Limiting the amplitude at ± 1 can be easily done by a diode/Z-diode pair connected between the output of the integrator and its summing junction. Avoiding a decreasing amplitude for small input values to the multipliers requires some additional positive feedback which is implemented by a potentiometer set to a very small value such as 0.005. This will result in a some typically negligible distortion of the sine function.

Figure 1 shows the analog computer setup for THOMAS' cyclically symmetric attractor. It requires nine integrators, six multipliers, 15 coefficient potentiometers, and six inverters. Figure 2 shows the actual setup on three coupled THE ANALOG THINGs. Setting the all important parameter b is a bit fiddly as the system is very sensitive to even small variations in b. Thus, it should be set to three decimal places on all three THATs.[6] Three typical results for different values of b are shown in figure 3.

[6]If one had three additional multipliers this could be simplified considerably.

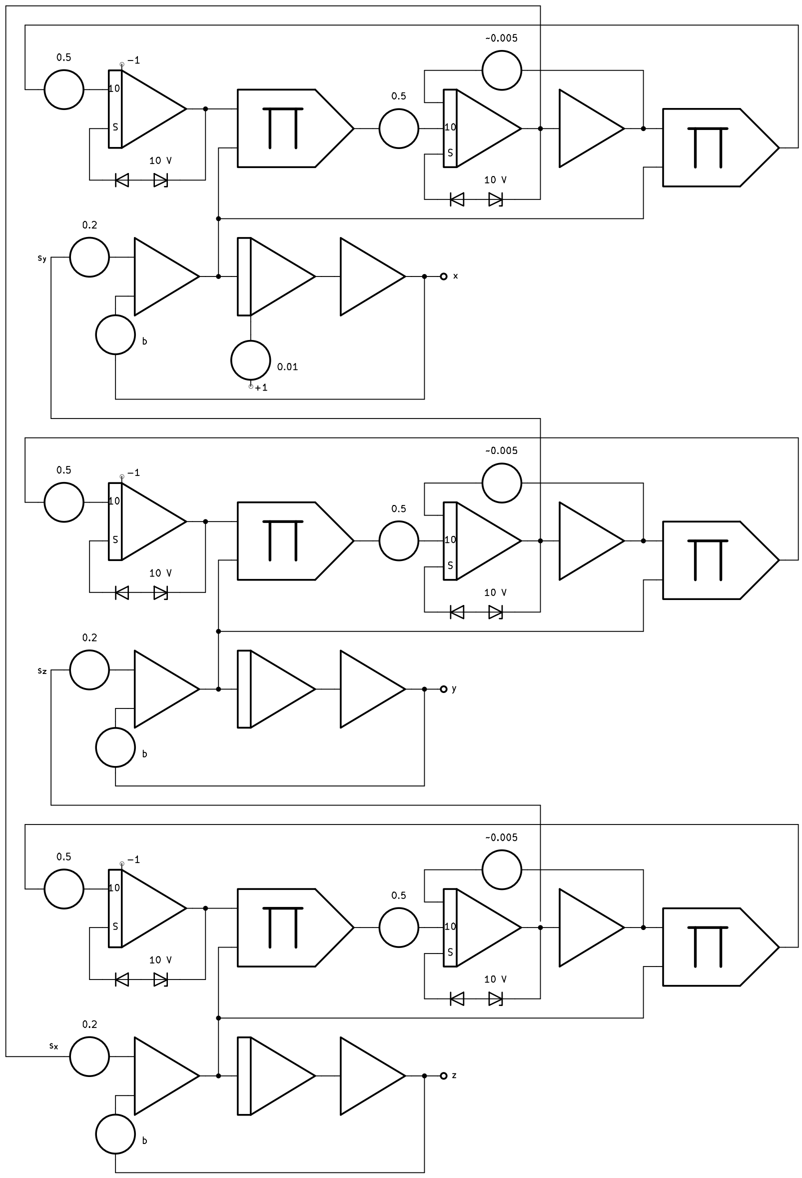

Figure 1: Analog computer program for equations (4), (5), (6), and (7), (8), (9)

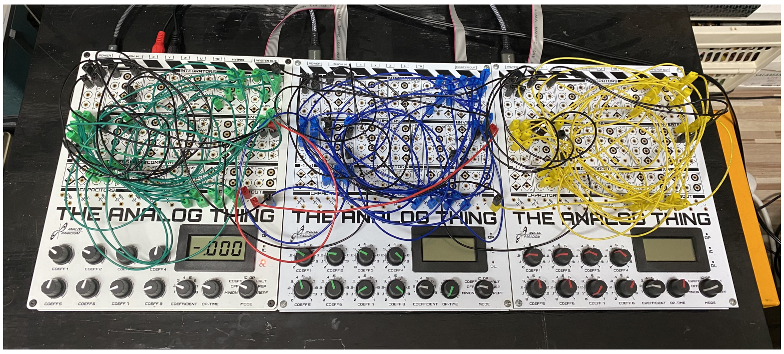

Figure 2: Analog computer setup for equations (4), (5), (6), and (7), (8), (9)

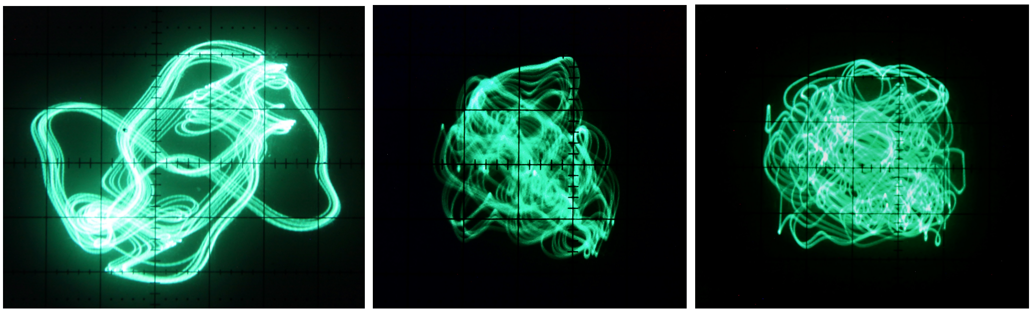

Figure 3: Typical results for different values of b

An obvious question with respect to this system is whether a simpler odd function instead of the sines would also yield a chaotic system. Such a system has also been described by THOMAS[7]:

$$\dot{x} = -bx + ay - y^3$$
$$\dot{y} = -by + az - z^3$$
$$\dot{z} = -bz + ax - x^3$$

A quick experiment shows that this system needs scaling factors $\lambda = \lambda_x = \lambda_y = \lambda_z = \frac{1}{2}$ resulting in the following system

$$\dot{x} = -bx + ay - 4y^3 \tag{10}$$
$$\dot{y} = -by + az - 4z^3 \tag{11}$$
$$\dot{z} = -bz + ax - 4x^3 \tag{12}$$

with parameters $a = 1.1$ and $b = 0.3$ and initial conditions $x(0) = 0.5$, $y(0) = z(0) = 0.$.

This system can be easily implemented on an analog computer and only requires three integrators, six multipliers, three inverters, and seven coefficient potentiometers as shown in figure 4. A long term exposure of the resulting oscilloscope display plotting x against y is shown in figure 5.

References

[BASIOS et al. 2018] VASILEIOS BASIOS, CHRIS G. ANTONOPOULOS, "Hyperchaos & Labyrinth chaos: Revisiting Thomas-Rössler systems", in *Journal of Theoretical Biology*, Volume 460, January 2019, pp. 153–159

[GILOI et al. 1963] WOLFGANG GILOI, RUDOLF LAUBER, *Analogrechnen*, Springer-Verlag, 1963

[7] See [THOMAS 1999, p. 1899].

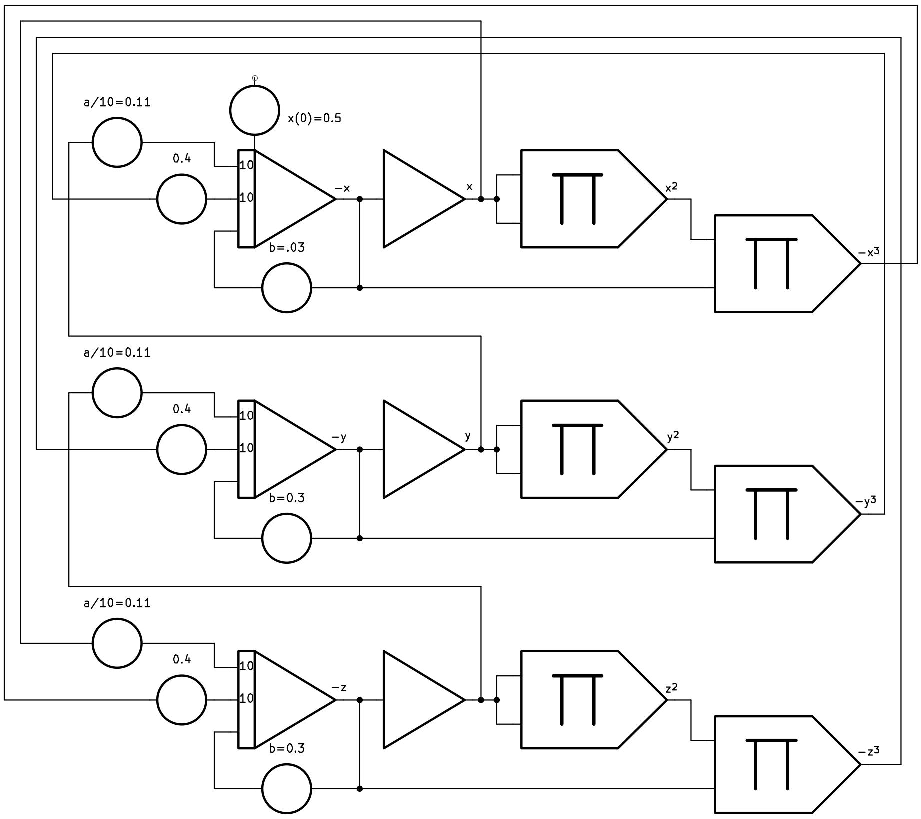

Figure 4: Analog computer program for equations (10), (11), and (12)

[Korn et al. 1964] Granino A. Korn, Theresa M. Korn, *Electronic Analog and Hybrid Computers*, McGraw-Hill Book Company, 1964

[Thomas 1999] R. Thomas, "Deterministic Chaos Seen in Terms of Feedback Circuits: Analysis, Synthesis 'Labyrinth Chaos'", in *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 09, 1999, p. 1889–1905

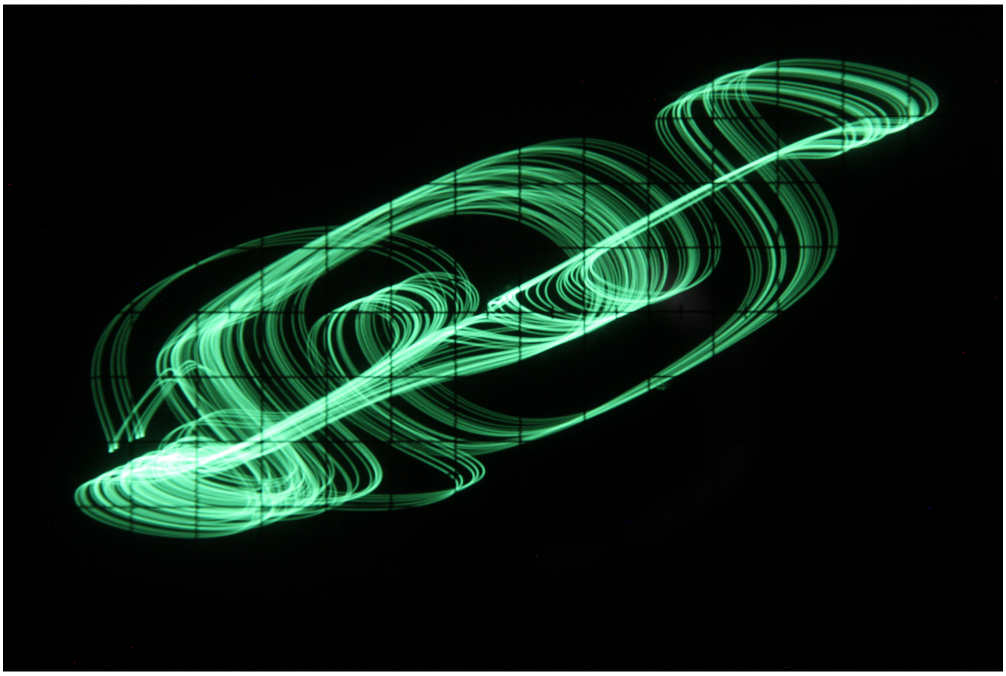

Figure 5: Phase space plot (x/y) of THOMAS' cubic attractor