Airflow around an airfoil[1]

1 Introduction

This application note deals with airflow around rotating cylinders and conformal mappings which transform the cross section of such a cylinder into a special type of airfoil.

2 Airflow around a rotating cylinder

Spinning objects such as golf balls or rotating cylinders experience a force which is due to the effect of their rotation in conjunction with the surrounding medium like air or water. This is known as MAGNUS *effect* after *Heinrich Gustav Magnus*[2], a German physicist who gave the first comprehensible explanation of the effect. In the following we will simulate the airflow around a rotating cylinder under the following assumptions:

- The cylinder with circular cross section is infinitely long.

- The airflow is always perpendicular to the cylinder axis. Furthermore it is stationary and conservative, i. e.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \text{ and } \frac{\mathrm{d}A}{\mathrm{d}t} = 0$$

  with v and A denoting the velocity and area of the airflow.

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[1]The author would like to thank MALCOLM BLUNDEN for proofreading.
[2]May 2nd, 1802–April 4th, 1870

- Furthermore the air is assumed to be incompressible and without friction.

Such an airflow is a potential flow and its velocity field can thus be described as the gradient of some complex function $f(z)$ called *velocity potential*. In the following the function

$$f(z) = v_0 \left(z e^{\mathrm{i}\varphi} + \frac{r^2 e^{-\mathrm{i}\varphi}}{z} \right) - \mathrm{i}\frac{\Gamma}{2\pi} \log(z)$$

will be used. Here r denotes the radius of the cylinder around which the air flows prior to its deformation into an airfoil by the conformal mapping, v_0 is the initial velocity of the air flow, φ the angle of attack and Γ the circulatory component.

2.1 Generating the cylinder cross section

First of all we need a circuit yielding the cross section of the rotating cylinder, i. e. a circle. This is implemented by solving the classic differential equation $\ddot{y} = -y$ as shown in figure 1.

The radius is set to about 0.4 by means of the parameter β yielding negative feedback and the initial condition of the first integrator which should match to avoid artifacts at the start of a simulation run. The coefficient α provides a very small positive feedback. It can be omitted if the radius does not decrease during a prolonged run. If necessary at all, it should be set to about 0.005, otherwise the sine/cosine signals will show distortion effects. The time constants of the two integrators are set to $k_0 = 10^3$ yielding an output frequency of about 1.2 kHz.
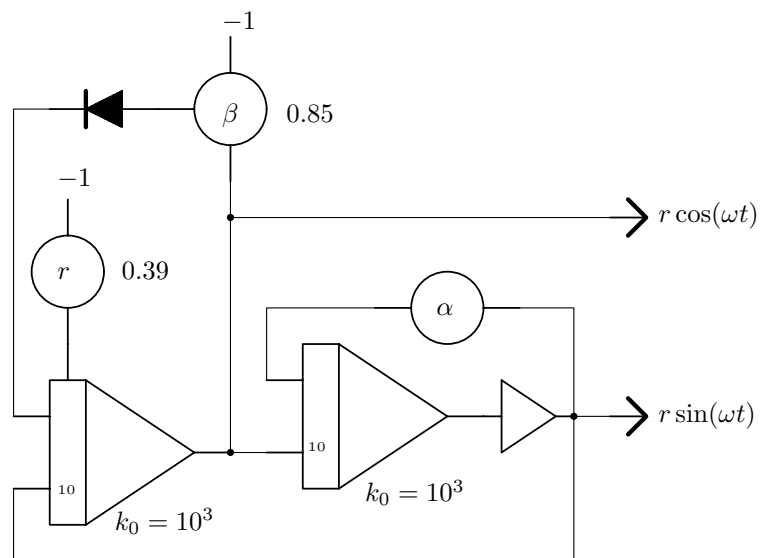
Figure 1: Generating a sine/cosine signal pair

2.2 Computing flowlines

The mechanization of this complex velocity potential is shown in figure 2. This circuit expects three parameters/inputs:

- The angle of attack, $\tan(\varphi)$, which can be set manually by the free potentiometer shown in the upper left,

- a value τ varying linearly from -1 to $+1$ during one computer run. This represents the actual movement of an air particle from left to right, i.e. its x-component.[3]

- Finally, the input denoted y_0 represents the distance of the flow line from the lower/upper surface of the airfoil. This value can be set either manually by a potentiometer connected to $+1$ or -1 or automatically as shown below.

The small capacitors shown on the free amplifiers are absolutely necessary to prevent the overall circuit from oscillating. A practical value for these capacitors is 47 pF. Figure 2 shows the corresponding computer setup.

Generating the time varying τ is easy as shown in figure 3: One integrator (time constant set to $k_0 = 10^2$ and a summer suffice to yield both, $-\tau$ and τ which will be used later.

As nice as it is to control the vertical (start) position of a flow line manually, it is advisable to generate a linearly time varying signal $-1 \leq y_0 \leq 1$ as shown in figure 4. As the computer is run in repetitive mode with an operation time of 20 ms, the integrator used to generate y_0

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[3]Note that the circuit actually requires $-\tau$ instead of τ.
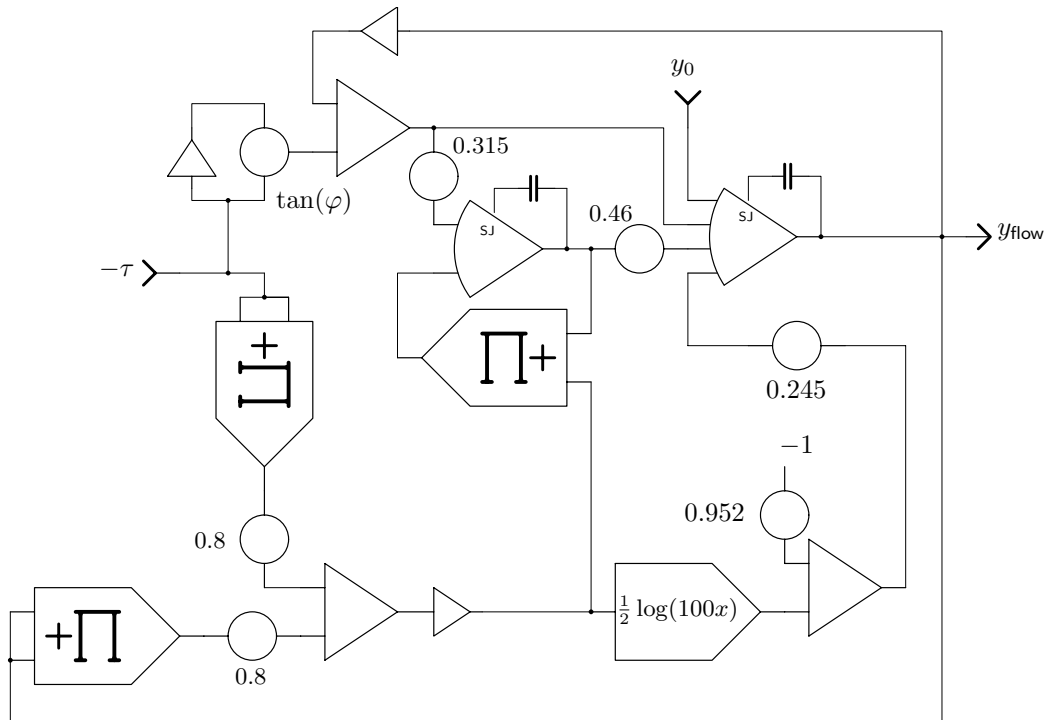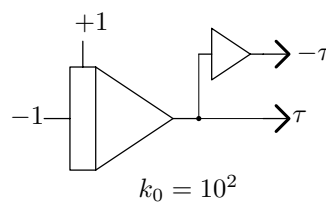
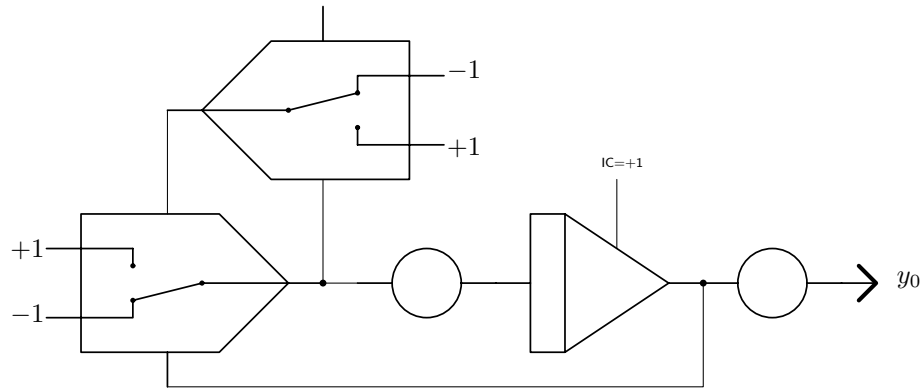Figure 2: Flowlines

Figure 3: Circuit yielding $\pm\tau$

Figure 4: Generating y_0

must be controlled externally. On an Analog Paradigm Model-1 this can be achieved by patching the IC-input of one of the two rightmost integrators of an INT4 module to $+1$. Using the two potentiometers shown in the figure the rate of change of y_0 and its range may be set.

2.3 Results

Putting all of this together and using a display with two (x, y)-inputs the airflow around a rotating cylinder can be visualized as shown in figure 5.

3 Conformal mapping

Using a so-called *conformal* or *angle-preserving* mapping it is now possible to transform the cylinder cross section into the shape of an airfoil.
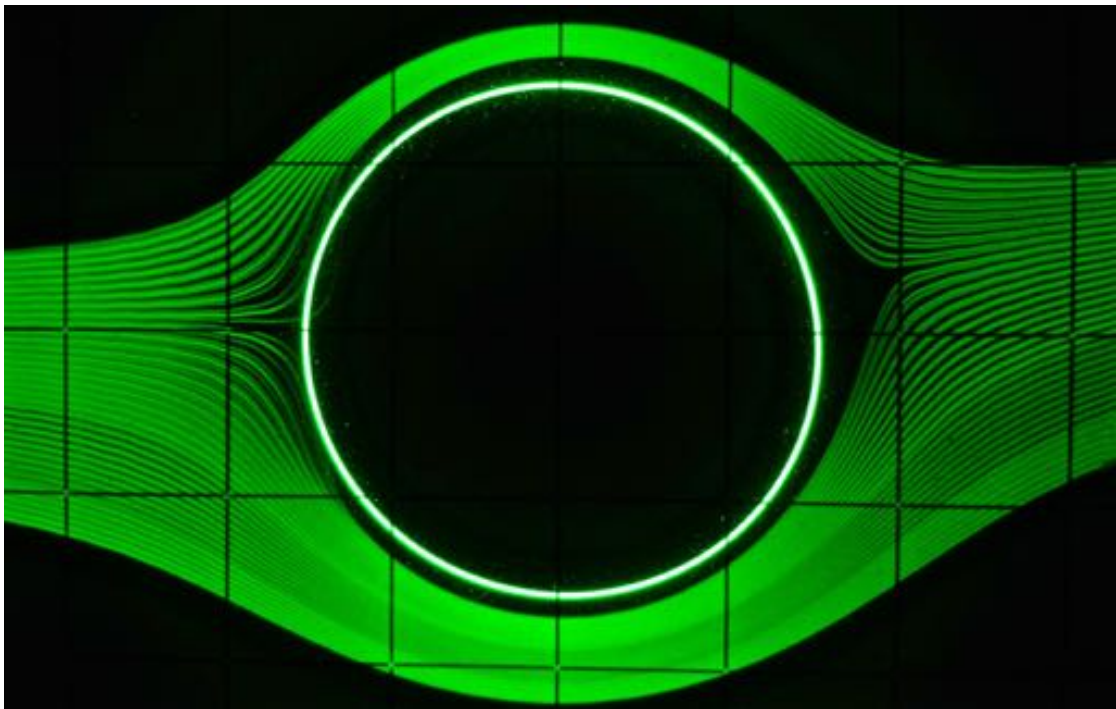
Figure 5: Flow around rotating cylinder

Applying the same mapping to the flow lines around the cylinder yields the corresponding flow lines around that particular airfoil.

Conformal mappings are functions generally defined on the complex plane $f : \mathbb{C} \to \mathbb{C}$ which preserve angles locally. One particularly demonstrative function which will be used in the following is the so-called JOUKOWSKY *transform*

$$z = \zeta + \frac{1}{\zeta},$$

developed by *Nikolai Zhukovsky*.[4] This function maps a (unit) circle into the shape or an airfoil, a so-called JOUKOWSKY *airfoil*.[5]

The following sections describing the actual implementation of such a mapping are based on [LOTZ],[6] [REUTTER et al. (1968)] and finally [SYDOW (1964)][p. 123].[7]

3.1   Conformal mapping

The actual conformal mapping as implemented is based on the transform

$$f(z) = (z - z_*) + \frac{\eta}{z - z_*}.$$

which is split into its real and imaginary parts yielding

$$u\left(x(t), y(t)\right) = (x(t) - x_*) + \frac{\eta^2 \left(x(t) - x_*\right)}{\left(x(t) - x_*\right)^2 + \left(y(t) - y_*\right)^2}$$

$$v\left(x(t), y(t)\right) = (y(t) - y_*) - \frac{\eta^2 \left(y(t) - y_*\right)}{\left(x(t) - x_*\right)^2 + \left(y(t) - y_*\right)^2}$$

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[4] January 17th, 1847–March 17th, 1921
[5] A noteworthy property of this class of air foils is a cusp at their trailing edge.
[6] The author would like to thank Mr. LOTZ for his support.
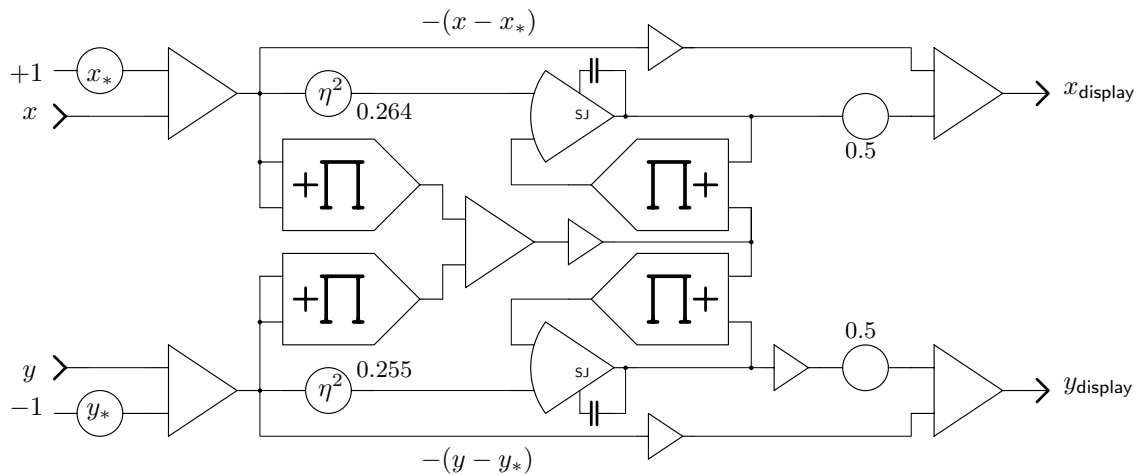[7] Especially [REUTTER et al. (1968)] contains a wealth of information on the general topic.

Figure 6: Conformal mapping

with

$$x(t) = r\cos(\omega t) \text{ and}$$
$$y(t) = r\sin(\omega t)$$

as yielded by the sub-circuit shown in figure 1. The parameters x_* and y_* define the shape of the resulting airfoil. Good values to start with are $x_* = 0.045$ and $y_* = 0.052$. Figure 6 shows the computer setup for the conformal mapping.

The two free amplifiers used to implement the division operations should have a small stabilizing capacitor connected between their respective output and the summing junction. As noted before, $47\,\text{pF}$ is a reasonable value for such capacitors.

3.2 Switching between profile and airflow

Since the conformal mapping is used twice (to transform the circle into a JOUKOWSKY airfoil and to transform the airflow around the cylinder into the airflow around that airfoil), the inputs of the circuit shown in figure 6 are switched between the outputs of the sub-circuit generating $r\sin(\omega t)$ and $r\cos(\omega t)$ and the circuit yielding the coordinates of an air particle flowing around the hypothetical cylinder.

This alternating of input values is implemented by the circuit shown in figure 7. The (electronic) switches are driven from one position to the other every other repetitive cycle of the computer. This requires a toggle flip-flop driven by the operate-control-line of the computer. The output of this flip-flop controls the switches.

3.3 Computer setup

The overall computer setup is quite complex and requires a rather unusual complement of computing elements, the majority of which are summers. All in all the following complement of components is required:

Component	Quantity	Component	Quantity
Summer	21	Integrator	4
Potentiometer	16	Comparator	4
Multiplier	7	$\frac{1}{2}\log(100x)$	1
Capacitor 47 pF	4	Diode	1
Toggle flip-flop	1		

The overall setup is shown in figure 8. It is based on Analogparadigm's Model-1 analog computer with an additional function generator yielding
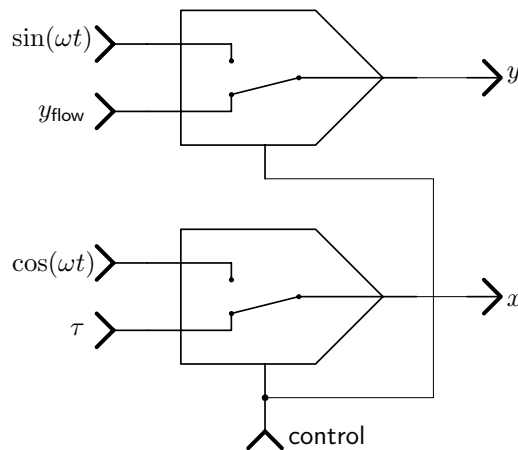
Figure 7: Switching between profile and airflow

$\frac{1}{2}\log(100x)$ which was built specially for this setup and an external flip-flop circuit for toggling the input switches of the conformal mapping subcircuit.[8]

4   Results

Figure 9 shows a typical output generated by this computer setup.

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[8]The log-function-generator is based on the LOG112 integrated circuit. A schematic can be provided on request.
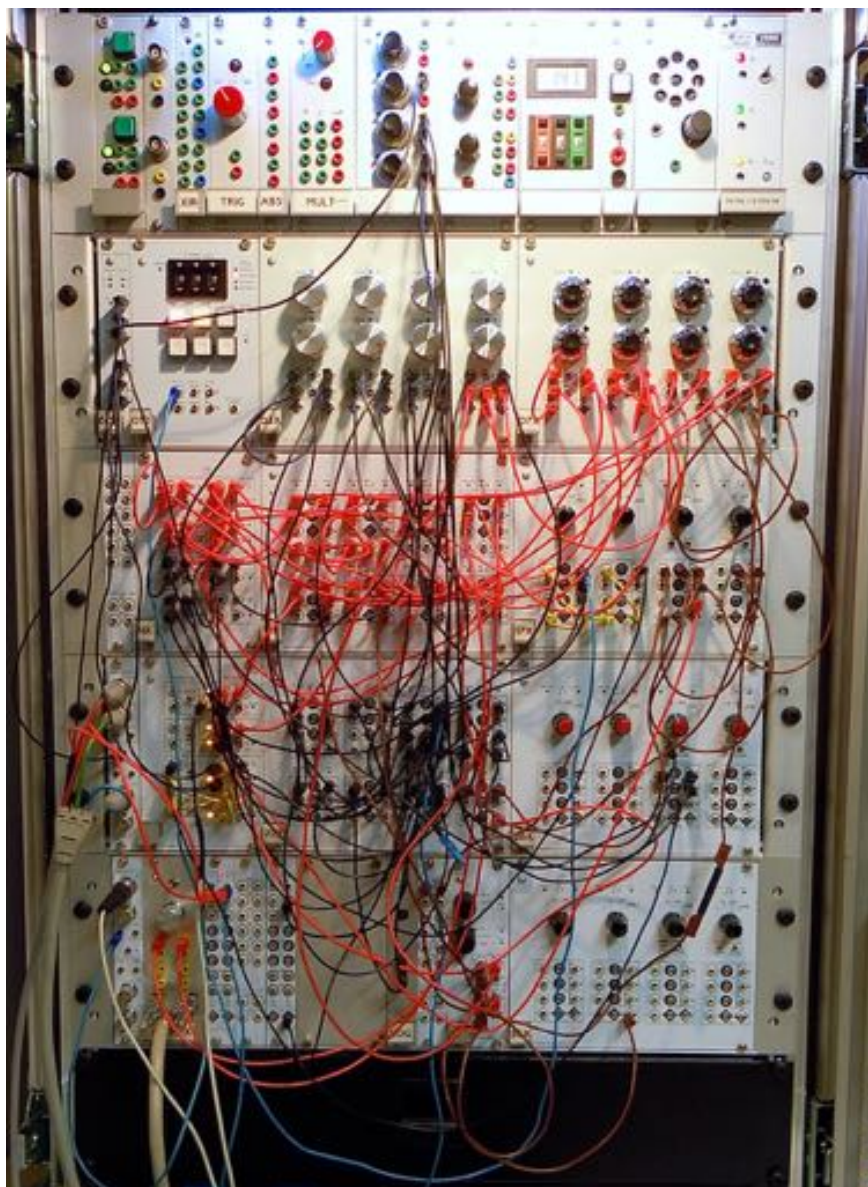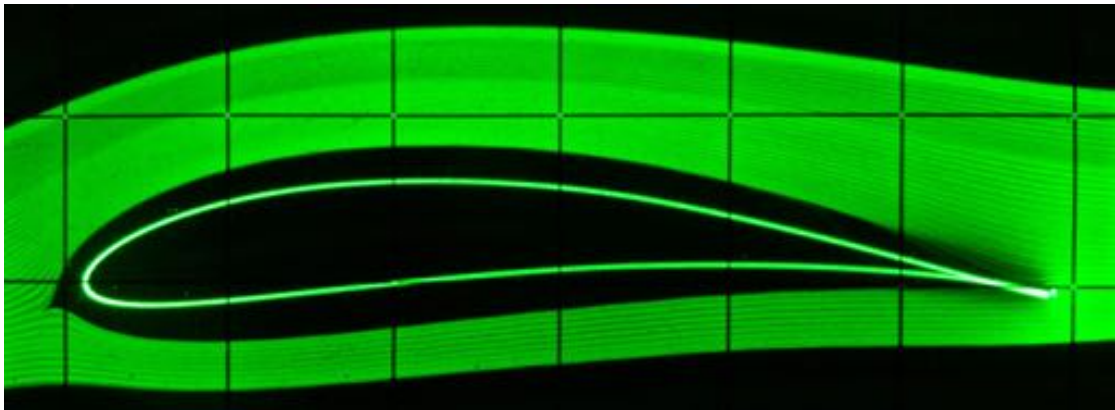
Figure 8: Overall computer setup

Figure 9: Flow around Joukowsky-profile

References

[Lotz] H. Lotz, *Darstellung von Tragflügeln und ihren Stromlinien mit einem Analogrechner*, AEG Telefunken, AAB 011 1069

[Reutter et al. (1968)] Fritz Reutter, Josef Neukirchen, Dietmar Sommer, *Herstellung konformer Abbildungen mit Hilfe des Analogrechners – Praktische Behandlung der Umströmung zweifach zusammenhängender Gebiete*, Westdeutscher Verlag, Köln und Opladen, 1968

[Sydow (1964)] Achim Sydow, *Programmierungstechnik für elektronische Analogrechner*, VEB Verlag Technik, Berlin, 1964